

## SOLUTION FOR SEPTEMBER 2026

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Prove that if  $a + b + c = abc$  and  $a, b, c$  are positive then

$$\pi = \tan^{-1}(a) + \tan^{-1}(b) + \tan^{-1}(c).$$

**Solution:**

First notice that if  $a + b + c = abc$  with  $a, b, c$  positive then  $ab \neq 1$  and  $c = (a + b)/(ab - 1)$ . (If  $ab = 1$  then  $a + b + c = abc = c$  thus  $a + b = 0$  but  $a > 0$  and  $b > 0$ . Therefore  $ab \neq 1$ ). Then since  $a + b + c = abc$  it follows that  $a + b = c(ab - 1)$ . Since  $ab \neq 1$  then  $c = (a + b)/(ab - 1)$ .

Next let  $a_1 = \tan^{-1}(a)$ ,  $b_1 = \tan^{-1}(b)$ ,  $c_1 = \tan^{-1}(c)$ .

Then

$$0 < a_1 < \pi/2, \quad 0 < b_1 < \pi/2, \quad 0 < c_1 < \pi/2$$

and so

$$0 < a_1 + b_1 + c_1 < 3\pi/2. \tag{1}$$

Also,  $\tan(a_1) = a$ ,  $\tan(b_1) = b$ ,  $\tan(c_1) = c$ . Then using the addition formula for  $\tan(x)$ , that  $\tan(x)$  is an odd function, and that  $c = (a + b)/(ab - 1)$  gives

$$\tan(a_1 + b_1) = \frac{\tan(a_1) + \tan(b_1)}{1 - \tan a_1 \tan b_1} = (a + b)/(1 - ab) = -c = -\tan(c_1) = \tan(-c_1).$$

Then since  $\tan$  is  $\pi$  periodic we see that  $a_1 + b_1 = -c_1 + k\pi$  for some integer  $k$  thus

$$a_1 + b_1 + c_1 = k\pi.$$

Then from (1) we see that we must have  $k = 1$  and thus

$$\tan^{-1}(a) + \tan^{-1}(b) + \tan^{-1}(c) = a_1 + b_1 + c_1 = \pi.$$

In particular, let  $a = 1, b = 2, c = 3$  then  $a + b + c = abc$  and thus

$$\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \pi.$$

□