SOLUTION FOR OCTOBER 2026

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Determine

$$\sum_{n=0}^{\infty} \frac{n!}{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)} = 1 + \frac{1}{3} + \frac{2!}{3 \cdot 5} + \frac{3!}{3 \cdot 5 \cdot 7} + \cdots + \frac{n!}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)} + \cdots .$$

HINT: First prove for $n \geq 1$ that

$$\int_0^{\frac{\pi}{2}} \cos^{2n+1}(x)\, dx = \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdot 9 \cdots (2n+1)} = 2^n \left(\frac{n!}{3 \cdot 5 \cdot 7 \cdot 9 \cdots (2n+1)} \right)$$

Solution

$$\sum_{n=0}^{\infty} \frac{n!}{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)} = \frac{\pi}{2}.$$

Proof We first establish the hint.

We denote

$$a_n = \int_0^{\frac{\pi}{2}} \cos^{2n+1}(x)\, dx. \tag{1}$$

We now want to show that

$$a_n = \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdot 9 \cdots (2n+1)} \text{ for } n \geq 1. \tag{2}$$

For $n = 1$ we have

$$a_1 = \int_0^{\frac{\pi}{2}} \cos^3(x)\, dx = \int_0^{\frac{\pi}{2}} \cos(x)(1 - \sin^2 x)\, dx = \left(\sin(x) - \frac{\sin^3(x)}{3} \right) \Big|_0^{\frac{\pi}{2}} = 1 - \frac{1}{3} = \frac{2}{3}.$$

This establishes (2) for $n = 1$.

Now by induction we assume the result (2) for n and establish it for $n + 1$. Then by (1) we have

$$a_{n+1} = \int_0^{\frac{\pi}{2}} \cos^{2(n+1)+1}(x)\, dx = \int_0^{\frac{\pi}{2}} \cos^{2n+1}(x) \cos^2(x)\, dx = \int_0^{\frac{\pi}{2}} \cos^{2n+1}(x) \left(1 - \sin^2(x)\right)\, dx$$

$$= a_n - \int_0^{\frac{\pi}{2}} \cos^{2n+1}(x) \sin^2(x)\, dx = a_n + \int_0^{\frac{\pi}{2}} \left(\frac{\cos^{2n+2}(x)}{2n+2} \right)' \sin(x)\, dx.$$

On the rightmost integral we integrate by parts and obtain

$$a_{n+1} = a_n - \int_0^{\frac{\pi}{2}} \frac{\cos^{2n+3}(x)}{2n+2}\,dx = a_n - \frac{a_{n+1}}{2n+2}.$$

Thus we see

$$a_{n+1} = \left(\frac{2n+2}{2n+3}\right) a_n.$$

Now using (2) this gives

$$a_{n+1} = \left(\frac{2n+2}{2n+3}\right) a_n = \frac{2\cdot 4\cdot 6\cdot 8\cdots(2n)(2(n+1))}{3\cdot 5\cdot 7\cdot 9\cdots(2n+1)(2n+3)}.$$

This then establishes (2) for $n \geq 1$.
Next, dividing both sides in the hint by 2^n and taking sums gives

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \int_0^{\frac{\pi}{2}} \cos^{2n+1}(x)\,dx = \sum_{n=1}^{\infty} \int_0^{\frac{\pi}{2}} \left(\frac{\cos^2(x)}{2}\right)^n \cos x\,dx = \sum_{n=1}^{\infty} \frac{n!}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}.$$

Since $0 \leq \frac{\cos^2(x)}{2} \leq \frac{1}{2} < 1$ we see that the infinite sum is geometric and converges uniformly.
Thus interchanging the integral and the sum is justified and we see

$$\sum_{n=1}^{\infty} \frac{n!}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)} = \sum_{n=1}^{\infty} \int_0^{\frac{\pi}{2}} \left(\frac{\cos^2(x)}{2}\right)^n \cos x\,dx = \int_0^{\frac{\pi}{2}} \sum_{n=1}^{\infty} \left(\frac{\cos^2(x)}{2}\right)^n \cos x\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos^2(x)}{2-\cos^2(x)} \cos x\,dx$$

$$= \int_0^{\frac{\pi}{2}} -\cos x\,dx + \int_0^{\frac{\pi}{2}} \frac{2\cos x}{1+\sin^2 x}\,dx = (-\sin x + 2\tan^{-1}(\sin x))\Big|_0^{\frac{\pi}{2}} = -1 + 2\tan^{-1}(1) = \frac{\pi}{2} - 1.$$

Thus

$$\sum_{n=0}^{\infty} \frac{n!}{1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)} = \frac{\pi}{2}.$$