

SOLUTION FOR DECEMBER 2024

A correct solution was submitted by:

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PROBLEM: Suppose f is continuous and $f(f(x)) = x$ for all x and $f(0) = 0$, $f(1) = 1$. Prove $f(x) = x$ on $[0, 1]$.

PROOF: First it follows that f is one-to-one for if $f(x_1) = f(x_2)$ then applying f to both sides gives $x_1 = f(f(x_1)) = f(f(x_2)) = x_2$. Thus f is one-to-one.

Now f cannot have a local maximum or local minimum on $(0, 1)$ because if so then f would not be one-to-one near that local extremum. Further since $f(0) = 0$ and $f(1) = 1$ and f is continuous then it follows that f must be increasing everywhere on $[0, 1]$.

We claim now that $f(x_1) = x_1$ for all x_1 in $(0, 1)$. So suppose not. Then there is an x_1 such that $f(x_1) \neq x_1$ and so without loss of generality let us suppose $f(x_1) > x_1$. Since f is increasing then applying f gives $x_1 = f(f(x_1)) > f(x_1)$ and thus $f(x_1) < x_1$ a contradiction. Thus $f(x_1) = x_1$ for all $x_1 \in (0, 1)$ and in fact on $[0, 1]$. $\square$